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2005 J. Phys. A: Math. Gen. 38 2497

(http://iopscience.iop.org/0305-4470/38/11/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 2497–2510 doi:10.1088/0305-4470/38/11/011

New two-dimensional quantum models partially solvable by the supersymmetrical approach

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Received 14 September 2004, in final form 30 December 2004
Published 2 March 2005
Online at stacks.iop.org/JPhysA/38/2497

Abstract
New solutions for second-order intertwining relations in two-dimensional SUSY QM are found via repeated use of the first-order supersymmetrical transformations with intermediate constant unitary rotation. Potentials obtained by this method—two-dimensional generalized Pöschl–Teller potentials—appear to be shape-invariant. The recently proposed method of SUSY—separation of variables—is implemented to obtain a part of their spectra, including the ground state. Explicit expressions for energy eigenvalues and corresponding normalizable eigenfunctions are given in an analytic form. Intertwining relations of higher orders are discussed.

PACS numbers: 03.65.−w, 03.65.Fd, 11.30.Pb

1. Introduction

The importance of each new exactly solvable model in one-dimensional (1D) quantum mechanics is well known, especially because the list of such models is quite small. The elegant modern approach used for the study and classification of these 'elite' models was provided by supersymmetrical quantum mechanics (SUSY QM) [1, 2], which is in essence an alternative formulation of the famous factorization method [3] in one-dimensional quantum mechanics. Furthermore, the introduction in the framework of SUSY QM of a new notion—the shape invariance [4, 2]—gave a novel, algebraic, tool to deal with such kinds of models. There are different ways of going beyond the scope of the standard Witten's SUSY QM in order to enlarge the class of involved models. The higher order SUSY QM (HSUSY QM), or equivalently, polynomial and N-fold SUSY QM [5, 6] as well as constructions for multidimensional coordinate spaces [7, 8] are among the most promising ones.

From the very beginning, after 1D SUSY QM was formulated by Witten [1], the question of finding the opportunity to generalize it for higher dimensions of space attracted considerable attention. A direct d-dimensional generalization was constructed in [8] by means of methods originating from SUSY quantum field theory. In this approach the superhamiltonian (of block-diagonal form) includes both scalar and matrix components and can be used to analyse different physical problems with matrix potentials [9].

In the particular case of two-dimensional space an alternative SUSY QM approach was proposed, which directly generalizes the HSUSY QM ideas, namely, the use of the SUSY intertwining relations with the second-order supercharges. This method avoids the appearance of matrix potentials and provides the intertwining of two scalar Schrödinger Hamiltonians. A large class of such intertwined Hamiltonians was found in [10–13].

In the framework of the latter approach two new methods for the study of the spectra and the (normalizable) eigenfunctions of two-dimensional quantum models were proposed recently in [14, 15]: SUSY-separation of variables and the two-dimensional shape invariance (see also the review-type paper [16]). The combination of both of them was explored to investigate a specific model—a generalized 2D Morse potential with three free parameters—which is not amenable to the conventional separation of variables. As a result, this model turned out to be *partially solvable*, i.e. only a part of the variety of its *normalizable* wavefunctions and corresponding values of energies were found *analytically*. Thus the transfer from one-dimensional to two-dimensional shape invariance was accompanied by the loss of complete solvability with only the partial one remaining. It is worth mentioning here that each of the 2D Hamiltonians involved in the second-order intertwining relation is integrable: the symmetry operator of the fourth order in derivatives was constructed explicitly in terms of supercharges [10, 12].

In this paper both approaches of two-dimensional SUSY QM—the direct two-dimensional generalization [7, 8] and the second-order construction [10–13]—will be used to construct and to investigate some new models, to which no standard separation of variables can be applied. Again SUSY-separation of variables turns out to be applicable to the model, providing a set of normalizable wavefunctions. This model, which is shown to be partially solvable, will be called a *2D-generalized Pöschl–Teller potential*.

As for the method of the two-dimensional shape invariance [14–16], the situation is more delicate. Though the considered model possesses the property of shape invariance, the corresponding solutions of the Schrödinger equation turned out to be unnormalizable.

The paper is organized as follows. In section 2 the known methods of 2D SUSY QM will be described briefly in order to simplify the comprehension of the new results. A new technique of searching for solutions of the two-dimensional second-order intertwining relations will be presented in section 3, and in particular, two-dimensional generalizations of Pöschl–Teller potentials will be constructed. In section 4 SUSY-separation of variables will be used to find a part of the spectrum of this model and analytical expressions for its wavefunctions, including the ground state. The peculiarities of shape invariance are also investigated. In section 5 an additional structure with two different superpartners for the same Hamiltonian is presented, new intertwining relations of fourth and sixth orders in derivatives are constructed (the last ones are shape-invariant).

2. Basics of 2D SUSY QM

2.1. 2D representation of SUSY algebra

The SUSY algebra of quantum mechanics is given by the following (anti)commutation relations [1]:

$$\{\hat{Q}^+, \hat{Q}^-\} = \hat{H}; \qquad \{\hat{Q}^+, \hat{Q}^+\} = \{\hat{Q}^-, \hat{Q}^-\} = 0; \qquad [\hat{Q}^\pm, \hat{H}] = 0. \quad (1)$$

In the case of two dimensions it can be realized [7, 8] by the following 4×4 matrix operators

$$\hat{H} = \begin{pmatrix} H^{(0)}(\vec{x}) & 0 & 0 \\ 0 & H^{(1)}_{ik}(\vec{x}) & 0 \\ 0 & 0 & H^{(2)}(\vec{x}) \end{pmatrix}, \qquad i,k = 1,2;$$
$$\hat{Q}^+ = (\hat{Q}^-)^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q_1^- & 0 & 0 & 0 \\ q_2^- & 0 & 0 & 0 \\ 0 & p_1^+ & p_2^+ & 0 \end{pmatrix}, \tag{2}$$

where two scalar Hamiltonians $H^{(0)}, H^{(2)}$ and one 2×2 matrix Hamiltonian $H^{(1)}_{ik}$ of Schrödinger type can be expressed in a quasifactorized form (compare with the factorized form in one-dimensional case [1, 2])

$$H^{(0)} = q_l^+ q_l^- = -\partial_l^2 + V^{(0)}(\vec{x}) = -\partial_l^2 + (\partial_l \chi(\vec{x}))^2 - \partial_l^2 \chi(\vec{x}),\ \partial_l^2 \equiv \partial_1^2 + \partial_2^2;$$
$$H^{(2)} = p_l^+ p_l^- = -\partial_l^2 + V^{(2)}(\vec{x}) = -\partial_l^2 + (\partial_l \chi(\vec{x}))^2 + \partial_l^2 \chi(\vec{x}); \tag{3}$$
$$H^{(1)}_{ik} = q_i^- q_k^+ + p_i^- p_k^+ = -\delta_{ik}\partial_l^2 + \delta_{ik}\big((\partial_l \chi(\vec{x}))^2 - \partial_l^2 \chi(\vec{x})\big) + 2\partial_i\partial_k \chi(\vec{x}),$$

with components of supercharges of first order in derivatives

$$q_l^\pm \equiv \mp\partial_l + \partial_l \chi(\vec{x}); \qquad p_l^\pm \equiv \epsilon_{lk} q_k^\mp, \tag{4}$$

where $\partial_i \equiv \partial/\partial x_i$ and summation over repeated indices is implied. Anticommutators in (1) produce the following intertwining relations for the component Hamiltonians $H^{(0)}$, $H^{(1)}_{ik}$, $H^{(2)}$ of the superhamiltonian (2)

$$H^{(0)} q_i^+ = q_k^+ H^{(1)}_{ki}; \qquad H^{(1)}_{ik} q_k^- = q_i^- H^{(0)};$$
$$H^{(1)}_{ik} p_k^- = p_i^- H^{(2)}; \qquad H^{(2)} p_i^+ = p_k^+ H^{(1)}_{ki}. \tag{5}$$

They connect the spectrum of the matrix Hamiltonian with spectra of two scalar ones. In general, $H^{(0)}$ and $H^{(2)}$ *are not* isospectral since $q_k^+ p_k^- \equiv 0$ due to (4).

2.2. *Second-order supercharges in 2D SUSY QM*

Two-dimensional SUSY QM models without any matrix Hamiltonians were constructed [10, 11, 16] by means of second-order supercharges

$$Q^+ = (Q^-)^\dagger = g_{ik}(\vec{x})\partial_i\partial_k + C_i(\vec{x})\partial_i + B(\vec{x}), \tag{6}$$

where g_{ik}, C_i, B are arbitrary real functions. Some particular solutions for two scalar Hamiltonians $H^{(0),(1)}$ which satisfy the intertwining relations

$$H^{(1)}(\vec{x}) Q^+ = Q^+ H^{(0)}(\vec{x}); \qquad H^{(0)}(\vec{x}) Q^- = Q^- H^{(1)}(\vec{x}) \tag{7}$$

were found. They both possess the symmetry operators $R^{(1,2)}$ of fourth order in derivatives [10, 12]

$$[R^{(i)}, H^{(i)}] = 0; \qquad i = 0,1; \qquad R^{(0)} = Q^- Q^+; \qquad R^{(1)} = Q^+ Q^-,$$

which are not, in general, polynomials of $H^{(i)}$.

In terms of the unknown functions $g_{ik}, C_i, B, V^{(0)}, V^{(1)}$ equation (7) has the form [10] of seven nonlinear partial differential equations, and its general solution is not known. To obtain particular solutions different ansätze for 'metrics' g_{ik} were used.

Only the choice of Laplacian (elliptic) metrics $g_{ik}(\vec{x}) = \mathrm{diag}(1,1)$ leads to Hamiltonians amenable to R-separation [17] of variables. All other choices of metrics give nontrivial results.

The case of Lorentz (hyperbolic) metrics $g_{ik} = \mathrm{diag}(1, -1)$ was investigated in papers [10–13]. In particular, the intertwining relations (7) were reduced to the pair of differential equations

$$\partial_1\partial_2 F = 0; \qquad \partial_-(C_- F) = -\partial_+(C_+ F), \tag{8}$$

where[1] $x_\pm = (x_1 \pm x_2)/\sqrt{2}$ and $C_{1,2}$ were proven to satisfy $C_\pm \equiv C_1 \mp C_2 \equiv C_\pm(\sqrt{2}x_\pm)$. Then potentials $V^{(0),(1)}$ and the supercharges Q^+ are expressed in terms of functions $C_\pm(\sqrt{2}x_\pm)$ and $F(\vec{x})$ which obviously can be written as $F = F_1(2x_1) + F_2(2x_2)$ according to (8):

$$V^{(0),(1)} = \mp\tfrac{1}{2}(C'_+(\sqrt{2}x_+) + C'_-(\sqrt{2}x_-)) + \tfrac{1}{8}\big(C_+^2(\sqrt{2}x_+) + C_-^2(\sqrt{2}x_-)\big) + \tfrac{1}{4}(F_2(2x_2) - F_1(2x_1)); \tag{9}$$

$$Q^+ = \big(\partial_1^2 - \partial_2^2\big) + C_1\partial_1 + C_2\partial_2 + B; \tag{10}$$

$$B = \tfrac{1}{4}\big(C_+(\sqrt{2}x_+)C_-(\sqrt{2}x_-) + F_1(2x_1) + F_2(2x_2)\big), \tag{11}$$

where the prime denotes the derivative of the function with respect to its argument. A list of particular solutions of (8) was obtained in [11–13]. In the next section we will obtain new solutions for the case of hyperbolic metrics.

3. New solutions for the Lorentz (hyperbolic) metrics

3.1. Intertwining of second order with reducible supercharges

Let us consider two superhamiltonians $\hat{H}$ and $\hat{\widetilde{H}}$ of 2D SUSY QM:

$$\hat{H} = \begin{pmatrix} H^{(0)}(\vec{x}) & 0 & 0 \\ 0 & H^{(1)}_{ik}(\vec{x}) & 0 \\ 0 & 0 & H^{(2)}(\vec{x}) \end{pmatrix}; \qquad \hat{\widetilde{H}} = \begin{pmatrix} \widetilde{H}^{(0)}(\vec{x}) & 0 & 0 \\ 0 & \widetilde{H}^{(1)}_{ik}(\vec{x}) & 0 \\ 0 & 0 & \widetilde{H}^{(2)}(\vec{x}) \end{pmatrix}, \tag{12}$$

with superpotentials $\chi(\vec{x})$ and $\widetilde{\chi}(\vec{x})$, correspondingly.

In addition, let $H^{(1)}_{ik}$ and $\widetilde{H}^{(1)}_{ik}$ be linked by a unitary 2×2 matrix transformation U:

$$U_{ik}\widetilde{H}^{(1)}_{kl} = H^{(1)}_{im}U_{ml}; \tag{13}$$

$$U = \alpha_0\sigma_0 + \mathrm{i}\,\overrightarrow{\alpha}\,\overrightarrow{\sigma}; \qquad \alpha_0^2 + \overrightarrow{\alpha}^2 = 1; \qquad \alpha_0, \alpha_i \in \mathbf{R}, \tag{14}$$

where σ_i are the Pauli matrices and σ_0 is the unit matrix.

Then (due to (5)) the scalar Hamiltonians $H^{(0)}$ and $\widetilde{H}^{(0)}$ can be included in the chain

$$H^{(0)} \overset{q_l^\pm}{\longleftrightarrow} H^{(1)}_{ik} \overset{U_{lm}}{\longleftrightarrow} \widetilde{H}^{(1)}_{ik} \overset{\tilde{q}_m^\mp}{\longleftrightarrow} \widetilde{H}^{(0)} \tag{15}$$

leading to the intertwining relations between a pair of scalar Hamiltonians

$$H^{(0)}Q^- = Q^-\widetilde{H}^{(0)}, \qquad Q^+H^{(0)} = \widetilde{H}^{(0)}Q^+ \tag{16}$$

with the second-order operators

$$Q^- = (Q^+)^\dagger = q_i^+U_{ik}\tilde{q}_k^-. \tag{17}$$

This intertwining operator is constructed from two first-order ones with the intermediate matrix transformation U_{ik}. Precisely, this matrix provides that such supercharges $Q^\pm$ are nontrivial and, contrary to subsection 2.1, can be naturally described as *reducible* (compare with the case of one-dimensional reducibility introduced in [5]).

[1] We use here the definition of $x_\pm$ slightly different from the analogous one in [11–13].

In contrast to the approach of [14] (see subsection 2.2), the first Hamiltonian in the chain (15) is quasifactorized according to equation (3). Therefore the solution of the corresponding Schrödinger equation with zero energy can be written as $\Psi_0^{(0)} \sim \exp(-\chi)$. Due to expression (17), $\exp(-\chi)$ is a zero mode of supercharge Q^+ as well. In general, until the specific form of $\chi(\vec{x})$ is chosen, the normalizability of this solution is not guaranteed. But in the concrete model (31) analysed below in subsection 3.3 the zero energy solution is *normalizable* due to asymptotic properties of $\chi(\vec{x})$ for corresponding ranges (46) of parameters.

Target Hamiltonians $H^{(0)}$ and $\widetilde{H}^{(0)}$ are expressed in terms of two unknown functions χ and $\tilde{\chi}$ (see (3)). To determine these functions one should substitute (3), (14) and (17) into (16). After some manipulations one obtains the system of equations for $\chi_{\pm} = (\chi \pm \tilde{\chi})/2$:

$$\alpha_3\Box\chi_- + 2\alpha_1\partial_1\partial_2\chi_- = 0; \qquad \alpha_1\Box\chi_+ - 2\alpha_3\partial_1\partial_2\chi_+ = 0; \tag{18}$$

$$\alpha_2\Box\chi_+ + 2\alpha_0\partial_1\partial_2\chi_- = 0; \qquad \alpha_0\Box\chi_- + 2\alpha_2\partial_1\partial_2\chi_+ = 0; \tag{19}$$

$$(\partial_k\chi_-)(\partial_k\chi_+) = 0, \tag{20}$$

where $\Box \equiv \partial_1^2 - \partial_2^2$. Equation (20) (which is equivalent to $(\partial_k\chi)^2 = (\partial_k\widetilde{\chi})^2$) can be used to simplify expressions (3) for the Hamiltonians $H^{(0)}$ and $\widetilde{H}^{(0)}$:

$$H^{(0)}, \widetilde{H}^{(0)} = -\partial_l^2 + \left((\partial_l\chi_+)^2 - \partial_l^2\chi_+\right) + \left((\partial_l\chi_-)^2 \mp \partial_l^2\chi_-\right). \tag{21}$$

Linear partial differential equations (18) and (19) can be easily solved, but the solution of the nonlinear equation (20) is a nontrivial problem.

3.2. The particular solutions of the intertwining relations

It can be shown that in the case when *all coefficients* α_i and α_0 in (14) do not vanish, potentials $V^{(0)}$ and $\widetilde{V}^{(0)}$ are fourth order polynomials in $x_{1,2}$ with some additional constraints for their coefficients. In this paper we will consider potentials beyond this rather narrow class, restricting ourselves to the *particular* case $\alpha_0 = \alpha_1 = \alpha_2 = 0$; $\alpha_3 \neq 0$, i.e. $U = \sigma_3$. Then the metrics of supercharges $Q^{\pm}$ is Lorentz, i.e. they belong to the class discussed in subsection 2.2.

For this case equations (18) and (19) read

$$\Box\chi_- = 0; \qquad \partial_1\partial_2\chi_+ = 0.$$

Their solution is

$$\chi_- = \mu_+(x_+) + \mu_-(x_-), \qquad \chi_+ = \mu_1(x_1) + \mu_2(x_2),$$

with $\mu_{1,2}$, $\mu_{\pm}$ being arbitrary functions. The last equation (20) takes the form

$$\mu_1'(x_1)\left[\mu_+'(x_+) + \mu_-'(x_-)\right] + \mu_2'(x_2)\left[\mu_+'(x_+) - \mu_-'(x_-)\right] = 0.$$

By substitutions $\phi \equiv \mu'$, it becomes purely functional (without derivatives) equation

$$\phi_1(x_1)[\phi_+(x_+) + \phi_-(x_-)] = -\phi_2(x_2)[\phi_+(x_+) - \phi_-(x_-)]. \tag{22}$$

The *general* solution of (22) is given in the appendix[2]. Some particular cases will be discussed in subsection 3.3.

The Hamiltonians (21) and intertwining operators (17) can be expressed in terms of ϕ as

$$\begin{aligned} V^{(0)}, \widetilde{V}^{(0)} &= \left(\phi_1^2(x_1) - \phi_1'(x_1)\right) + \left(\phi_2^2(x_2) - \phi_2'(x_2)\right) + \left(\phi_+^2(x_+) \mp \phi_+'(x_+)\right) \\ &\quad + (\phi_-^2(x_-) \mp \phi_-'(x_-)), \\ Q^{\pm} &= \partial_1^2 - \partial_2^2 \pm \sqrt{2}(\phi_+(x_+) + \phi_-(x_-))\partial_1 \mp \sqrt{2}(\phi_+(x_+) - \phi_-(x_-))\partial_2 \\ &\quad - \left(\phi_1^2(x_1) - \phi_1'(x_1)\right) + \left(\phi_2^2(x_2) - \phi_2'(x_2)\right) + 2\phi_+(x_+)\phi_-(x_-). \end{aligned} \tag{23}$$

[2] It was derived by D N Nishnianidze (private communication).

By rearrangement of terms equation (22) can be rewritten as

$$\phi_+(x_+)[\phi_1(x_1)+\phi_2(x_2)] = -\phi_-(x_-)[\phi_1(x_1)-\phi_2(x_2)], \tag{24}$$

i.e. in a form similar to the initial equation (22). This means that (22) possesses the symmetry property (which will be called S_1-symmetry in the subsequent text): if $\{\phi_1(x_1), \phi_2(x_2), \phi_+(x_+), \phi_-(x_-)\}$ is a solution, then $\{\phi_+(x_1), \phi_-(x_2), \phi_1(x_+), \phi_2(x_-)\}$ is also a solution. Let us mention one more discrete symmetry of (20), S_2 symmetry: $\{\phi_1(x_1), \phi_2(x_2), \phi_+(x_+), \phi_-(x_-)\} \longrightarrow \{\phi_1(x_1), -\phi_2(x_2), \phi_+^{-1}(x_+), \phi_-^{-1}(x_-)\}$. The S_1-symmetry produces another supersymmetrical model

$$\begin{aligned}\mathcal{V}^{(0)}, \widetilde{\mathcal{V}}^{(0)} &= \left(\phi_1^2(x_+) \mp \phi_1'(x_+)\right) + \left(\phi_2^2(x_-) \mp \phi_2'(x_-)\right) + \left(\phi_+^2(x_1) - \phi_+'(x_1)\right)\\ &\quad + \left(\phi_-^2(x_2) - \phi_-'(x_2)\right),\\ \mathcal{Q}^{\pm} &= \partial_1^2 - \partial_2^2 \pm \sqrt{2}(\phi_1(x_+)+\phi_2(x_-))\partial_1 \mp \sqrt{2}(\phi_1(x_+)-\phi_2(x_-))\partial_2\\ &\quad - \left(\phi_+^2(x_1) - \phi_+'(x_1)\right) + (\phi_-^2(x_2) - \phi_-'(x_2)) + 2\phi_1(x_+)\phi_2(x_-).\end{aligned} \tag{25}$$

Below both forms (23) and (25) will be explored.

To compare the new notation of this section with those of [11–13] (see subsection 2.2.) one can use the following relations:

$$C_{\pm}(\sqrt{2}x_{\pm}) = 2\sqrt{2}\phi_{\pm}(x_{\pm}); \qquad F_{1,2}(2x_{1,2}) = \mp 4\left(\phi_{1,2}^2(x_{1,2}) - \phi_{1,2}'(x_{1,2})\right). \tag{26}$$

3.3. Nonperiodical solutions for potentials $V^{(0)}, \widetilde{V}^{(0)}$

From equation (A8) one can conclude that for an arbitrary choice of the parameters a, b, c the functions $\phi_{1,2}$ are expressed in terms of elliptic (Jacobi or Weierstrass) functions [18] (volume 3). In this paper we restrict ourselves to the limiting cases, for which the potentials are not periodical (the models with periodicity properties in $x_{1,2}$ will be studied elsewhere).

The integral in the rhs in (A8) is an elementary function only if either the some of coefficients a, b, c are zero or the quadratic polynomial is a full square. There are two families of solutions of (22), with members interconnected by symmetries S_1 and S_2 (and their combinations):

(i) $\phi_1(x) = \phi_2(x) = A/x; \qquad \phi_+ = \phi_- = B/x \quad (A, B = \text{const}).$

The multiparticle potentials of this type were found in [19] to be quasi-exactly solvable [20]. All other members of this family allow the separation of variables.

(ii) $\phi_1(x) = \phi_2(x) = M(\delta_+ e^{\alpha x} + \delta_- e^{-\alpha x});$

$$\phi_+(x) = -L\frac{\delta_+ e^{\alpha x/\sqrt{2}} - \delta_- e^{-\alpha x/\sqrt{2}}}{\delta_+ e^{\alpha x/\sqrt{2}} + \delta_- e^{-\alpha x/\sqrt{2}}}; \qquad \phi_-(x) = L\coth(\alpha x/\sqrt{2}). \tag{27}$$

For the particular case $\delta_- = 0$ one has

$$\begin{aligned}V^{(0)}, \widetilde{V}^{(0)} &= (B^2 e^{-2\alpha x_1} + B\alpha e^{-\alpha x_1}) + (B^2 e^{-2\alpha x_2} + B\alpha e^{-\alpha x_2})\\ &\quad + 4A^2 + A(2A \mp \alpha)\left[\sinh\left(\frac{\alpha}{2}(x_1 - x_2)\right)\right]^{-2}\end{aligned} \tag{28}$$

with two new constants A, B instead of M, L, δ_+. These potentials (up to translations in $x_{1,2}$) were analysed in [14] and were found to be shape-invariant.

Another particular case $\delta_+ = -\delta_-$ for (27) after using symmetries and redefinition of parameters gives

$$\begin{aligned}\phi_1 &= -\phi_2 = \frac{A}{\sinh\sqrt{2}\alpha x}\\ \phi_+ &= \phi_- = B\tanh\alpha x.\end{aligned} \tag{29}$$

The corresponding potentials and intertwining operators for this model due to (23) are

$$\begin{aligned}
V^{(0)}, \widetilde{V}^{(0)} &= \left(B^2 - \frac{B(B\pm\alpha)}{\cosh^2\left(\frac{\alpha}{\sqrt{2}}(x_1+x_2)\right)}\right) + \left(B^2 - \frac{B(B\pm\alpha)}{\cosh^2\left(\frac{\alpha}{\sqrt{2}}(x_1-x_2)\right)}\right)\\
&\quad + A\left[\frac{A-\sqrt{2}\alpha\cosh(\sqrt{2}\alpha x_1)}{\sinh^2(\sqrt{2}\alpha x_1)} + \frac{A+\sqrt{2}\alpha\cosh(\sqrt{2}\alpha x_2)}{\sinh^2(\sqrt{2}\alpha x_2)}\right],\\
Q^{\pm} &= \partial_1^2 - \partial_2^2 \pm \sqrt{2}B\left[\tanh\left(\frac{\alpha}{\sqrt{2}}(x_1+x_2)\right) + \tanh\left(\frac{\alpha}{\sqrt{2}}(x_1-x_2)\right)\right]\partial_1\\
&\quad \mp \sqrt{2}B\left[\tanh\left(\frac{\alpha}{\sqrt{2}}(x_1+x_2)\right) - \tanh\left(\frac{\alpha}{\sqrt{2}}(x_1-x_2)\right)\right]\partial_2\\
&\quad - A\left[\frac{A-\sqrt{2}\alpha\cosh(\sqrt{2}\alpha x_1)}{\sinh^2(\sqrt{2}\alpha x_1)} - \frac{A+\sqrt{2}\alpha\cosh(\sqrt{2}\alpha x_2)}{\sinh^2(\sqrt{2}\alpha x_2)}\right]\\
&\quad + 2B^2\tanh\left(\frac{\alpha}{\sqrt{2}}(x_1+x_2)\right)\tanh\left(\frac{\alpha}{\sqrt{2}}(x_1-x_2)\right).
\end{aligned} \tag{30}$$

Solution (25), obtained by the discrete symmetry S_1, is

$$\begin{aligned}
\mathcal{V}^{(0)}, \widetilde{\mathcal{V}}^{(0)} &= \left(B^2 - \frac{B(B+\alpha)}{\cosh^2(\alpha x_1)}\right) + \left(B^2 - \frac{B(B+\alpha)}{\cosh^2(\alpha x_2)}\right)\\
&\quad + A\left[\frac{A\mp\sqrt{2}\alpha\cosh(\alpha(x_1+x_2))}{\sinh^2(\alpha(x_1+x_2))} + \frac{A\pm\sqrt{2}\alpha\cosh(\alpha(x_1-x_2))}{\sinh^2(\alpha(x_1-x_2))}\right]\\
\mathcal{Q}^{\pm} &= \partial_1^2 - \partial_2^2 \pm \sqrt{2}A\left[\frac{1}{\sinh(\alpha(x_1+x_2))} + \frac{1}{\sinh(\alpha(x_1-x_2))}\right]\partial_1\\
&\quad \mp \sqrt{2}A\left[\frac{1}{\sinh(\alpha(x_1+x_2))} - \frac{1}{\sinh(\alpha(x_1-x_2))}\right]\partial_2\\
&\quad - \left[B^2 - \frac{B(B+\alpha)}{\cosh^2(\alpha x_1)}\right] + \left[B^2 - \frac{B(B+\alpha)}{\cosh^2(\alpha x_2)}\right]\\
&\quad + \frac{2A^2}{\sinh(\alpha(x_1+x_2))\sinh(\alpha(x_1-x_2))}.
\end{aligned} \tag{31}$$

Both potentials (30) and (31) can be treated as superpositions of two one-dimensional Pöschl–Teller terms plus a singular term (so we will refer to them as *2D-generalized Pöschl–Teller potentials*). Each of them possesses a term which prevents application of the conventional method of separation of variables to determine their eigenfunctions and eigenvalues. Meanwhile, a part of the spectrum and corresponding eigenfunctions will be found by the method of SUSY-*separation of variables* (see [14, 16]) in the next section[3].

4. Partial solvability of 2D-generalized Pöschl–Teller potentials

4.1. SUSY-separation of variables

Because Hamiltonians with potentials (31) are intertwined by operators $\mathcal{Q}^{\pm}$ with the Lorentz metrics, we shall briefly recall the general method for searching eigenvalues and eigenfunctions proposed in [14]. From intertwining relations $\mathcal{Q}^{+}\mathcal{H}^{(0)} = \widetilde{\mathcal{H}}^{(0)}\mathcal{Q}^{+}$ (where $\mathcal{H}^{(0)} = -\partial_l^2 + \mathcal{V}^{(0)}$ and

[3] Other members of the same family (S_2- and S_2S_1-symmetric to (29)) can be treated analogously.

$\widetilde{\mathcal{H}}^{(0)} = -\partial_l^2 + \widetilde{\mathcal{V}}^{(0)}$) one obtains the fact that the subspace of zero-modes[4] of the supercharge $\mathcal{Q}^+$

$$\mathcal{Q}^+\vec{\Omega}(\vec{x}) = 0, \tag{32}$$

is closed under the action of $\mathcal{H}^{(0)}$:

$$\mathcal{H}^{(0)}\vec{\Omega}(\vec{x}) = \hat{C}\vec{\Omega}(\vec{x}) \tag{33}$$

with some constant matrix $\hat{C}$.

To determine the eigenvalues E_k and eigenfunctions $\Psi_k(\vec{x})$ of $\mathcal{H}^{(0)}$ one needs (see more details in [14]) a matrix $\hat{B}$, which satisfies the matrix equation $\hat{B}\hat{C} = \hat{\Lambda}\hat{B}$ with an unknown yet diagonal matrix $\hat{\Lambda} = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_N)$. Then the matrix $\hat{B}$ transforms zero-modes Ω_n into wavefunctions Ψ_n.

Operator $\mathcal{Q}^+$ belongs to type (10). For this type of supercharges problem (32) permits the conventional separation of variables in $\mathcal{Q}^+$ by means of 'gauge' transformation, which separates variables in the supercharge

$$\mathfrak{q}^+ = \mathrm{e}^{-\kappa(\vec{x})}\mathcal{Q}^+\,\mathrm{e}^{\kappa(\vec{x})} = \partial_1^2 - \partial_2^2 + \tfrac{1}{4}(F_1(2x_1) + F_2(2x_2)), \tag{34}$$

$$h(\vec{x}) \equiv \mathrm{e}^{-\kappa(\vec{x})}\mathcal{H}^{(0)}(\vec{x})\,\mathrm{e}^{\kappa(\vec{x})} = -\partial_1^2 - \partial_2^2 + C_1(\vec{x})\partial_1 - C_2(\vec{x})\partial_2 - \tfrac{1}{4}F_1(2x_1) + \tfrac{1}{4}F_2(2x_2),$$

$$\kappa(\vec{x}) \equiv -\frac{\sqrt{2}}{4}\left[\int C_+(\sqrt{2}x_+)\,\mathrm{d}x_+ + \int C_-(\sqrt{2}x_-)\,\mathrm{d}x_-\right], \tag{35}$$

$$\omega_n(\vec{x}) = \mathrm{e}^{-\kappa(\vec{x})}\Omega_n(\vec{x}).$$

Then the zero modes $\omega_n(\vec{x})$ of $\mathfrak{q}^+$ can be written as products $\omega_n(\vec{x}) = \eta_n(x_1)\rho_n(x_2)$, where ρ_n and η_n are eigenfunctions of the one-dimensional Schrödinger equations with 'potentials' $\left(\mp\frac{1}{4}F_{1,2}(2x_{1,2})\right)$, correspondingly (see (34)), and common eigenvalues (constants of separation) ϵ_n.

It is obvious that $h\vec{\omega} = \hat{C}\vec{\omega}$ with the same matrix $\hat{C}$ as in (33). The simplest way to find $\hat{C}$ is to calculate the rhs of (35), which can be rewritten as

$$h\omega_n = [2\epsilon_n + C_1(\vec{x})\partial_1 - C_2(\vec{x})\partial_2]\omega_n. \tag{36}$$

As a result, after construction of the matrix $\hat{B}$ one will obtain part of the spectrum E_k and the corresponding wavefunctions $\Psi_k(\vec{x})$.

4.1.1. Calculation of $\hat{C}$. This general method, proposed in [14] and used there successfully to investigate the 2D Morse potential, can be applied to the pair $\mathcal{V}^{(0)}$, $\widetilde{\mathcal{V}}^{(0)}$ as well. In this case both one-dimensional equations for multipliers $\eta_n(x_1)$ and $\rho_n(x_2)$ have the same 'potentials'—*one-dimensional* Pöschl–Teller potentials—being exactly solvable

$$\left(-\partial_1^2 + B^2 - B(B+\alpha)\cosh^{-2}(\alpha x_1)\right)\eta_n(x_1) = \epsilon_n\eta_n(x_1) \tag{37}$$

and a similar equation for $\rho_n(x_2)$. By the change of variable $\xi \equiv \tanh\alpha x_1$, equation (37) can be reduced to the generalized Legendre equation [21]

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[(1-\xi^2)\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right] + \left[s(s+1) - \left(s^2 - \frac{\epsilon}{\alpha^2}\right)\frac{1}{1-\xi^2}\right]\eta = 0,$$

where $s = B/\alpha$. To have the finite solution at $\xi = -1$ the condition $\sqrt{(B^2-\epsilon)/\alpha^2} - s = -n$, $n \in \mathbb{N}$ must be satisfied. It gives a discrete set of values for the separation constant ϵ

$$\epsilon_n = \alpha^2 n(2s - n),$$

[4] Here we suppose that $(N+1)$ normalizable zero-modes $\Omega_n(\vec{x})$ are known, and $\vec{\Omega}(\vec{x})$ is a column vector with components $\Omega_n(\vec{x})$, $n = 0, 1, \ldots, N$.

for $n < s$. Corresponding functions (up to normalization factors) are $\eta_n = P_s^{s-n}(\xi)$, where $P_\nu^\mu(x)$ are the (generalized) Legendre functions. Thus, one achieves the expression for ω_n

$$\omega_n = P_s^{s-n}(\xi_1)P_s^{s-n}(\xi_2).$$

with $\xi_i = \tanh\alpha x_i$, $i = 1, 2$.

The next step in calculating the eigenfunctions is evaluating the rhs of (36):

$$h\omega_n(\vec{x}) = 2\alpha^2 n(2s-n)\omega_n - 2\sqrt{2}A\alpha(2s-n)(n+1)\frac{\left(1-\xi_1^2\right)^{1/2}\left(1-\xi_2^2\right)^{1/2}}{\left(\xi_1^2-\xi_2^2\right)}\Pi(n,s;\xi_1,\xi_2), \tag{38}$$

where we have used the shorthand notation

$$\Pi(n,s;\xi_1,\xi_2) = \xi_2\left(\xi_1^2-1\right)^{1/2}P_s^{s-n-1}(\xi_1)P_s^{s-n}(\xi_2) - \xi_1\left(\xi_2^2-1\right)^{1/2}P_s^{s-n}(\xi_1)P_s^{s-n-1}(\xi_2).$$

Our goal is to represent the rhs of (38) as a linear combination of ω_k. For this purpose we use the recurrent formula for Legendre functions (see [18], vol 1, page 161, equation (1))

$$(z^2-1)^{1/2}P_\nu^{\mu+2}(z) + 2(\mu+1)zP_n^{\mu+1}u(z) = (z^2-1)^{1/2}(\nu-\mu)(\mu+\nu+1)P_\nu^\mu.$$

Applying it twice to $\Pi(n,s;\xi_1;\xi_2)$ one obtains the following recurrent formula:

$$\Pi(n,s;\xi_1,\xi_2) = \frac{1}{(n+1)n(2s-n)(2s-n+1)} \times\left[\frac{2(s-n+1)\left(\xi_1^2-\xi_2^2\right)}{\left(1-\xi_1^2\right)^{1/2}\left(1-\xi_2^2\right)^{1/2}}\omega_{n-1} + \Pi(n-2,s;\xi_1,\xi_2)\right]. \tag{39}$$

To stop this procedure at $n = 0$, one has to consider $s \in \mathbb{N}$. In this case the Legendre functions turn into the associate Legendre polynomials, for which $P_n^m(z) \equiv 0$ for $m > n$. So, applying (39) several times

$$\Pi(n,s;\xi_1,\xi_2) = \frac{\xi_1^2-\xi_2^2}{\left(1-\xi_1^2\right)^{1/2}\left(1-\xi_2^2\right)^{1/2}}\sum_{k=0}^{n}a_{nk}\omega_k$$

with constants a_{nk}. The matrix elements c_{nk} of matrix $\hat{C}$ are

$$c_{nk} = 2\alpha^2 n(2s-n)\delta_{nk} - 2\sqrt{2}A\alpha(2s-n)(n+1)a_{nk}; \tag{40}$$

$$a_{nk} = \begin{cases} 0, & k \geqslant n; \\ 0, & k = n-2m-2;\ m = 0,1,2,\ldots \\ 2(s-k)\frac{(k-1)!(2s-n-1)!}{(n+1)!(2s-k)!}, & k = n-2m-1;\ m = 0,1,2,\ldots. \end{cases} \tag{41}$$

4.1.2. Calculation of eigenfunctions. Matrix $\hat{C}$ for the model (31) appeared to be triangular, and hence its eigenvalues coincide with the diagonal elements

$$E_k = c_{kk} = 2\alpha^2 k(2s-k). \tag{42}$$

This formula gives $E_0 = 0$, demonstrating that the zero energy solution of $\mathcal{H}^{(0)}$ is a zero mode of $\mathcal{Q}^+$ as well: $\Psi_0 \sim \exp(-\chi(\vec{x}))$.

In order to avoid zeros on the diagonal of $\hat{C}$ one can shift Hamiltonians by a constant γ. This transformation does not destroy the intertwining relations (16) and changes $\hat{C}$ as follows:

$c_{ik} \to c_{ik} + \gamma\delta_{ik}$. This new $\hat{C}$ can be diagonalized by the method presented in [14]. Namely, the formal solution for the matrix elements of $\hat{B}$ reads

$$b_{m,p} = b_{m,N-m}\left[\sum_{l=1}^{N-p-1}(\tau^{(m)})^l\right]_{N-m,p}, \tag{43}$$

where $(N+1)$ matrices $\tau^{(m)}$ are defined by

$$\tau^{(m)}_{n,k} \equiv \frac{c_{n,k}}{c_{N-m,N-m} - c_{k,k}}$$

and label (m) has values $m = 0, 1, \ldots, N$. In equation (43) the repeated index $N - m$ is *not* summed over, and (to avoid misunderstanding) $\tau^{(m)^l}$ means the lth power of matrix $\tau^{(m)}$. Thus one obtains the recipe for the construction of eigenfunctions for $\mathcal{H}^{(0)}$ in (31):

$$\Psi_{N-n}(\vec{x}) = \sum_{k=0}^{N} b_{n,k}\Omega_k(\vec{x}). \tag{44}$$

Formula (43) gives us the opportunity to express an element $b_{m,p}$ by means of the $\tau^{(m)}$ matrices and an arbitrary element $b_{m,N-m}$ on the crossed diagonal. This last element can be fixed by the normalization condition for Ψ_{N-m}. The reason for the 'inverted' numeration of Ψ in (44) is to make Ψ_k dependent only on Ω_l; $l = 0, 1, \ldots, k$. In particular, $\Psi_0 \sim \Omega_0$. So, applying the method of SUSY-separation of variables to $\mathcal{H}^{(0)}$, $\widetilde{\mathcal{H}}^{(0)}$, one obtains a set of eigenvalues and eigenfunctions for $\mathcal{H}^{(0)}$.

Keeping in mind that $s = B/\alpha > 0$, we can restrict ourselves to $B > 0, \alpha > 0$. The conditions of normalizability of Ω_n (and therefore of Ψ_n) for all ω_n can be derived from the explicit expressions

$$\Omega_n = \omega_n \exp\kappa = \left(\frac{(\cosh(\alpha(x_1+x_2)) - 1)(\sinh(\alpha(x_1-x_2)))}{\sinh(\alpha(x_1+x_2))(\cosh(\alpha(x_1-x_2)) - 1)}\right)^{A/(\sqrt{2}\alpha)} P_s^{s-n}(\xi_1)P_s^{s-n}(\xi_2). \tag{45}$$

The constraint is $-\frac{1}{\sqrt{2}} < \frac{A}{\alpha} < \frac{1}{\sqrt{2}}$. The condition $A > 0$ keeps the strength of attractive singularities of both superpartners $\mathcal{V}^{(0)}$, $\widetilde{\mathcal{V}}^{(0)}$ at $x_\pm \to 0$ not exceeding the standard bound $-1\big/\big(4x_\pm^2\big)$. The resulting range of parameters for both Ω_n and $\widetilde{\Omega}_n$ is

$$\alpha > 0; \qquad B > 0; \qquad \frac{B}{\alpha} \in \mathbb{N}; \qquad 0 < A < \frac{\alpha}{\sqrt{2}}. \tag{46}$$

At first it seems that the energy eigenvalues (42), which were built above from the analysis of the zero-modes of $\mathcal{Q}^+$, should be absent in the spectrum of its superpartner $\widetilde{\mathcal{H}}^{(0)}$, since the corresponding eigenfunctions are annihilated by $\mathcal{Q}^+$. However, the whole procedure of SUSY-separation of variables can also be implemented for the spectral problem for $\widetilde{\mathcal{H}}^{(0)}$ by suitably replacing (32) with $\mathcal{Q}^-\tilde{\tilde{\Omega}} = 0$. Since $\mathcal{Q}^-$ and $\mathcal{Q}^+$ differ only by sign in front of the first derivatives (see (25)), one should use the 'gauge' transformation with $\exp(-\kappa(\vec{x}))$. In this case one will obtain the same equations (37), as for problem (32). Then the zero-modes of $\mathcal{Q}^-$ can be written as

$$\widetilde{\Omega}_n(\vec{x}) = \exp(-\kappa(\vec{x}))\omega_n(\vec{x}) = \exp(-2\kappa(\vec{x}))\Omega_n(\vec{x}),$$

and the corresponding matrix $\hat{C}$ is again triangular. To be more precise, it is the same as (40) and (41) up to the sign of the last term in (40). Therefore, its eigenvalues, i.e. values of energy for $\widetilde{\mathcal{H}}^{(0)}$, coincide with (42). One can check that the eigenfunctions are normalizable in the same range of parameters (46). Thus the obtained part of the spectra of superpartners $\mathcal{H}^{(0)}$ and $\widetilde{\mathcal{H}}^{(0)}$ *totally* coincide. In a certain sense this result is similar to one of the variants of the second-order intertwining in 1D HSUSY QM [5]: the equal number of bosonic and fermionic zero modes *does not* signal the spontaneous breaking of the supersymmetry.

4.2. The method of shape-invariance

Shape-invariance [4, 2, 14] is an additional property of intertwined superpartner Hamiltonians which gives the opportunity to determine their spectra algebraically. Namely, if both Hamiltonians depend on some extra parameter (or set of parameters) a, this property reads

$$\widetilde{H}(a_0) = H(a_1) + \mathcal{R}(a_0), \tag{47}$$

where $a_1 = f(a_0)$ is another value of the parameter, and $\mathcal{R}(a_0)$ does not depend on $\vec{x}$, i.e. $\widetilde{H}$ has *the same* (up to an additive constant) shape as H, but with another set of parameters.

Let us assume that we know some eigenfunction $\Psi^{(0)}$ of H and the corresponding eigenvalue $E^{(0)}$ in some range of the parameter a. Starting from

$$H(a_1)\Psi^{(0)}(a_1) = E^{(0)}(a_1)\Psi^{(0)}(a_1) \tag{48}$$

and employing (47), one obtains

$$\widetilde{H}(a_0)\Psi^{(0)}(a_1) = (E^{(0)}(a_1) + \mathcal{R}(a_0))\Psi^{(0)}(a_1). \tag{49}$$

Using intertwining relations (16) in (49),

$$H(a_0)[Q^-(a_0)\Psi^{(0)}(a_1)] = (E^{(0)}(a_1) + \mathcal{R}(a_0))[Q^-(a_0)\Psi^{(0)}(a_1)]. \tag{50}$$

This means that $H(a_0)$ has the eigenvalue $E^{(1)}(a_0) = E^{(0)}(a_1) + \mathcal{R}(a_0)$ with the wavefunction $\Psi^{(1)}(a_0) = Q^-(a_0)\Psi^{(0)}(a_1)$ (its normalizability is not guaranteed). Starting from (48) with the parameter $a_2 = f(f(a_0))$ and repeating the described procedure twice one can find

$$H(a_0)[Q^-(a_0)Q^-(a_1)\Psi^{(0)}(a_2)] = (E^{(0)}(a_2) + \mathcal{R}(a_1) + \mathcal{R}(a_0))[Q^-(a_0)Q^-(a_1)\Psi^{(0)}(a_2)], \tag{51}$$

which gives one more point $(\Psi^{(2)}(a_0), E^{(2)}(a_0))$ in the spectrum of $H(a_0)$. The general formulae are

$$\Psi^{(n)}(a_0) = Q^-(a_0)Q^-(a_1)\cdots Q^-(a_{n-1})\Psi^{(0)}(a_n), \tag{52}$$

$$E^{(n)}(a_0) = E^{(0)}(a_n) + \sum_{k=0}^{n-1}\mathcal{R}(a_k). \tag{53}$$

So, we have constructed a 'shape-invariance chain' of eigenfunctions starting from the one given. The natural idea is to combine this method with SUSY-separation of variables (if the Hamiltonians possess shape-invariance): having $(N+1)$ eigenfunction from SUSY-separation, we use each of them to start the above described shape-invariance chain. This procedure was implemented for the generalized 2D Morse potential (28) in [14].

For the 2D-generalized Pöschl–Teller potential the situation becomes more complicated. Indeed, the Hamiltonians $H^{(0)}$, $\widetilde{H}^{(0)}$ with potentials (30) are shape-invariant:

$$\widetilde{H}^{(0)}(\vec{x}; B, \alpha) = H^{(0)}(\vec{x}; B-\alpha, \alpha) + 2(B^2 - (B-\alpha)^2)), \tag{54}$$

where, in the notation introduced above, $a_0 \equiv B$; $a_1 = f(a_0) \equiv B - \alpha$. But, contrary to the model [14], the method of SUSY-separation of variables *does not* work here, since the zero modes ω_n are unnormalizable for all values of the parameter A.

This obstacle can be overcome by exploring the relation between systems (30) and (31):

$$H^{(0)}(x_1, x_2) = \mathcal{H}^{(0)}(x_+, x_-), \tag{55}$$

where in the rhs arguments x_1, x_2 are substituted by x_+, x_-. Because a part of the spectrum (and the eigenfunctions) of $\mathcal{H}^{(0)}$ was found by the method of SUSY-separation of variables (subsection 4.1), one can use relation (35) to obtain the corresponding part of the spectrum (and the eigenfunctions) for $H^{(0)}$. Then one can use these data to start shape invariance chains for the system $H^{(0)}$, $\widetilde{H}^{(0)}$ according to (52) with operators Q^-. For example, starting from the first zero mode one obtains

$$\Psi^{(n)}(x_1, x_2; B) = Q^-(x_1, x_2; B) \cdots Q^-(x_1, x_2; B - (n-1)\alpha)\Omega_0(x_+, x_-; B - n\alpha). \qquad (56)$$

The general formula for the spectrum can be obtained from (53), where the $E^{(0)}$ for each chain are taken from (42):

$$E_{mn} = 2\alpha^2[m(2s - 2n - m) + n(2s - n)] = 2\alpha^2(m + n)[2s - (m + n)], \qquad (57)$$

where $0 < m < s$ corresponds to the number of the chain (number of eigenfunction constructed by SUSY-separation), and $0 < n < s$ in order to keep positive all of $\mathcal{R}(a_k), k = 0, \ldots, (N-1)$, since the ground state energies $E^{(0)}(a_k) = 0$. Comparing (57) with (42) one will find that these points of the spectrum coincide exactly ($k \equiv m + n$). But at closer examination *all* 'wavefunctions' of the form (56) with $n \geqslant 1$ are unnormalizable due to the singular behaviour of the supercharges (30) at $x_{1,2} \to 0$. Thus the seeming $(k+1)$-fold degeneracy of kth energy level in (57) is spurious since only one of the solutions of the Schrödinger equation (namely, the linear combination of zero-modes Ω_n) is normalizable. Therefore, in contrast to the method of SUSY-separation of variables, the method of 2D shape-invariance is *ineffective* to give *normalizable* shape-invariance chains of wavefunctions for the 2D Pöschl–Teller potential.

5. Two-dimensional intertwining relations of more than second order

In this section we will imply equivalence of $H^{(0)}$ and $\mathcal{H}^{(0)}$ up to a change of variables for the new construction. Due to (35), the intertwining relation $\mathcal{H}^{(0)}\mathcal{Q}^- = \mathcal{Q}^-\widetilde{\mathcal{H}}^{(0)}$ can be rewritten as

$$H^{(0)}\widetilde{\mathcal{Q}}^- = \widetilde{\mathcal{Q}}^-\breve{\mathcal{H}}^{(0)},$$

where $\widetilde{\mathcal{Q}}^\pm(x_1, x_2) = \mathcal{Q}^\pm(x_+, x_-)$ and $\breve{\mathcal{H}}^{(0)}(x_1, x_2) = \widetilde{\mathcal{H}}^{(0)}(x_+, x_-)$. Comparing it with intertwining relations for the pair $(H^{(0)}, \widetilde{H}^{(0)})$, one can conclude that $H^{(0)}$ has two *different* superpartners

$$\breve{\mathcal{H}}^{(0)} \overset{\widetilde{\mathcal{Q}}^\pm}{\longleftrightarrow} H^{(0)} \overset{Q^\mp}{\longleftrightarrow} \widetilde{H}^{(0)} \qquad (58)$$

intertwined by *different* supercharges. Therefore, the Hamiltonians $\breve{\mathcal{H}}^{(0)}$ and $\widetilde{H}^{(0)}$ can be considered as superpartners intertwined by the *fourth order* operators $\widetilde{\mathcal{Q}}^\pm Q^\mp$. This pair does not obey the shape-invariance property.

Because, the Hamiltonian $\widetilde{H}^{(0)}$ is shape-invariant (47), one can develop the construction (58)

$$\breve{\mathcal{H}}^{(0)}(a_0) \overset{\widetilde{\mathcal{Q}}^\pm(a_0)}{\longleftrightarrow} H^{(0)}(a_0) \overset{Q^\mp(a_0)}{\longleftrightarrow} \widetilde{H}^{(0)}(a_0) = H^{(0)}(a_1) + \mathcal{R}(a_0) \overset{\widetilde{\mathcal{Q}}^\mp(a_1)}{\longleftrightarrow} \breve{\mathcal{H}}^{(0)}(a_1) + \mathcal{R}(a_0), \qquad (59)$$

where $a_0 = B, a_1 = B - \alpha$ (see subsection 4.2). The outermost operators in (59) are intertwined by *sixth order* supercharges according to

$$\breve{\mathcal{H}}^{(0)}(a_0)[\widetilde{\mathcal{Q}}^+(a_0)Q^-(a_0)\widetilde{\mathcal{Q}}^-(a_1)] = [\widetilde{\mathcal{Q}}^+(a_0)Q^-(a_0)\widetilde{\mathcal{Q}}^-(a_1)][\breve{\mathcal{H}}^{(0)}(a_1) + \mathcal{R}(a_0)] \qquad (60)$$

and contrary to the previous, fourth order, case, obey the shape-invariance property. Thus one can continue the construction of the spectrum of $\breve{\mathcal{H}}^{(0)}$.

Acknowledgments

The authors are indebted to A A Andrianov for useful discussions, and especially to D N Nishnianidze for careful reading of the manuscript and for derivation of the general solution of equation (22). PV is grateful to International Centre of Fundamental Physics in Moscow and the non-profit foundation 'Dynasty' for financial support. This work was partially supported by the grant No. 05-01-01090 of the Russian Foundation for Basic Research.

Appendix. The general solution of the functional equation

Applying operator $\left(\partial_1^2 - \partial_2^2\right)$ to both sides of (22), one has

$$2\left(\frac{\tilde{\phi}_1'(x_1)}{\tilde{\phi}_1(x_1)}\partial_1 - \frac{\phi_2'(x_2)}{\phi_2(x_2)}\partial_2\right)(\phi_-(x_-) - \phi_+(x_+)) = \left(\frac{\phi_2''(x_2)}{\phi_2(x_2)} - \frac{\phi_1''(x_1)}{\phi_1(x_1)}\right)(\phi_-(x_-) - \phi_+(x_+)), \tag{A1}$$

where the notation $\tilde{\phi}_1(x_1) = 1/\phi_1(x_1)$ was used. The general solution of (A1) is

$$\begin{aligned}\phi_-(x_-) - \phi_+(x_+) &= (\tilde{\phi}_1'(x_1)\phi_2'(x_2))^{-1/2}\Lambda\left(\int \frac{\tilde{\phi}_1(x_1)}{\tilde{\phi}_1'(x_1)}\,\mathrm{d}x_1 + \int \frac{\phi_2(x_2)}{\phi_2'(x_2)}\,\mathrm{d}x_2\right)\\ &= \phi_1(x_1)(\phi_1'(x_1)\phi_2'(x_2))^{-1/2}\Lambda\left(\int \frac{\phi_1(x_1)}{\phi_1'(x_1)}\,\mathrm{d}x_1 - \int \frac{\phi_2(x_2)}{\phi_2'(x_2)}\,\mathrm{d}x_2\right),\end{aligned} \tag{A2}$$

where Λ is an arbitrary function. The corresponding expression for $(\phi_-(x_-) + \phi_+(x_+))$ can be obtained using initial equation (22):

$$\phi_-(x_-) + \phi_+(x_+) = \phi_2(x_2)(\phi_1'(x_1)\phi_2'(x_2))^{-1/2}\Lambda\left(\int \frac{\phi_1(x_1)}{\phi_1'(x_1)}\,\mathrm{d}x_1 - \int \frac{\phi_2(x_2)}{\phi_2'(x_2)}\,\mathrm{d}x_2\right). \tag{A3}$$

Expressions for $\phi_\pm$ in terms of Λ should depend on the proper argument, i.e. $\partial_\pm\phi_\mp = 0$, leading to additional constraints for the function Λ:

$$\frac{1}{2}\left(\frac{\phi_2''(x_2)}{\phi_2(x_2)\phi_2'(x_2)} + \frac{\phi_1''(x_1)}{\phi_1(x_1)\phi_1'(x_1)}\right)\Lambda = \left(\frac{1}{\phi_1'(x_1)} - \frac{1}{\phi_2'(x_2)}\right)\Lambda', \tag{A4}$$

$$\left(\phi_1'(x_1) + \phi_2'(x_2) - \frac{\phi_1(x_1)\phi_1''(x_1)}{2\phi_1'(x_1)} - \frac{\phi_2(x_2)\phi_2''(x_2)}{2\phi_2'(x_2)}\right)\Lambda = \left(\frac{\phi_2^2(x_2)}{\phi_2'(x_2)} - \frac{\phi_1^2(x_1)}{\phi_1'(x_1)}\right)\Lambda'. \tag{A5}$$

Its trivial solution $\Lambda \equiv 0$ gives $\phi_+ = \phi_- = 0$, $\phi_{1,2}$-arbitrary, and the potentials (23) and (25) are amenable to separation of variables.

Otherwise one can exclude Λ from (A4) and (A5):

$$\frac{\phi_1''(x_1)\phi_2^2(x_2)}{\phi_1(x_1)} - \frac{\phi_2''(x_2)\phi_1^2(x_1)}{\phi_2(x_2)} = 2\phi_2'^2(x_2) - 2\phi_1'^2(x_1) + \phi_1(x_1)\phi_1''(x_1) - \phi_2(x_2)\phi_2''(x_2). \tag{A6}$$

Though there is no separation of variables in (A6), it will appear after applying the operator $\partial_1\partial_2$, so that

$$\frac{(\phi_1''/\phi_1)'}{\left(\phi_1^2\right)'} = \frac{(\phi_2''/\phi_2)'}{\left(\phi_2^2\right)'} \equiv 2a = \text{const}. \tag{A7}$$

Integrating, multiplying by $\phi'_{1,2}$, integrating again and taking into account (A6), one obtains the general solution of (22) in the form

$$\phi'^2_{1,2} = a\phi^4_{1,2} + b\phi^2_{1,2} + c; \qquad x = \pm \int \frac{\mathrm{d}\phi_1}{\sqrt{a\phi_1^4 + b\phi_1^2 + c}}; \qquad b, c = \text{const.} \tag{A8}$$

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